
Modern approaches to quantum gravity

Homework 8

Fall 2025

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1. Unitarity bounds

The generators of the conformal algebra can be represented as follows

$$\hat{P}_\mu = -i\partial_\mu \,, \qquad \hat{L}_{\mu\nu} = -i(x_\mu \partial_\nu - x_\nu \partial_\mu) \tag{1}$$

$$\hat{D} = -x \cdot \partial \,, \qquad \hat{K}_\mu = i(2x_\mu x \cdot \partial - x^2 \partial_\mu) \tag{2}$$

(a) In a unitary representation, there is a positive-definite inner product such that

$$\hat{D}^\dagger = D \,, \qquad \hat{K}_\mu^\dagger = P_\mu \,, \qquad \hat{L}_{\mu\nu}^\dagger = L_{\mu\nu} \tag{3}$$

Recalling that for a vector state, we have

$$\hat{L}_{\mu\nu} |\mathcal{O}^\alpha\rangle = (M_{\mu\nu})^\alpha{}_\beta |\mathcal{O}^\beta\rangle \,, \qquad (M_{\mu\nu})^\alpha{}_\beta = i(\eta_{\nu\beta}\delta^\alpha_\mu - \eta_{\mu\beta}\delta^\alpha_\nu) \,, \tag{4}$$

show that a vector primary state contained in a unitary representation must have dimension larger or equal to $d-1$. Show that when the bound is saturated, the state is created by a conserved current. Hint: compute the norm of $P_\mu|\mathcal{O}^\mu\rangle$.

(b) Verify that the operator

$$\hat{C} = \hat{D}^2 - \frac{1}{2}(\hat{K}_\mu \hat{P}^\mu + \hat{P}_\mu \hat{K}^\mu) + \frac{1}{2}\hat{L}_{\mu\nu}\hat{L}^{\mu\nu} \tag{5}$$

is a Casimir of the conformal algebra (i.e. it commutes with all its generators). Determine its value for a scalar and a vector primary state.

(c) Generalize the previous two questions for symmetric traceless primary state of spin $\ell \geq 2$ $|\mathcal{O}^{\alpha_1 \cdots \alpha_\ell}\rangle$. Recall that for such a state,

$$\hat{L}_{\mu\nu}|\mathcal{O}^{\alpha_1\cdots\alpha_\ell}\rangle = \sum_{i=1}^{\ell} (M_{\mu\nu})^{\alpha_i}{}_\beta |\mathcal{O}^{\alpha_1\cdots\alpha_{i-1}\beta\alpha_{i+1}\cdots\alpha_\ell}\rangle \tag{6}$$

You should find that the dimension of such a state in a unitary theory must be greater or equal to $d-2+\ell$.

2. Radial quantization of a free scalar field

(a) Consider the action for a free massless scalar field in Euclidean signature:

$$S_E = \int d^d x \frac{1}{2}(\partial_\mu \phi)^2, \quad g_{\mu\nu} = \delta_{\mu\nu}. \tag{7}$$

We want to quantize the theory on a surface of constant radius $r = (x_\mu x^\mu)^{1/2}$. Different "constant time" surfaces are related by the scale transformation $r \to e^\lambda r$. If we define $\sigma = \ln r$, time evolution is realized by the shift: $\sigma \to \sigma + \lambda$. The coordinate σ will be our time variable.

(b) Change coordinates in the action to (σ, θ^i), where θ^i collectively denotes the $d-1$ angular variables in spherical coordinates — for instance, when $d = 3$, $\theta^i = (\theta, \phi)$. Show that the action becomes

$$S_E = \int d\sigma \int d\Omega \frac{1}{2}\left((\partial_\sigma \chi)^2 + \partial_i \chi \partial^i \chi + \left(\frac{d-2}{2}\right)^2 \chi^2\right), \quad \chi = e^{\frac{d-2}{2}\sigma}\phi. \tag{8}$$

Here, $d\Omega$ is the area element on a sphere of unit radius (for instance, when $d = 3$, $d\Omega = \sin\theta \, d\theta \, d\phi$). Disregard, as usual, total derivatives.

(c) The usual Legendre transform almost gives the following Hamiltonian:

$$H = \int d\Omega \frac{1}{2}\left(-(\partial_\sigma \chi)^2 + \partial_i \chi \partial^i \chi + \left(\frac{d-2}{2}\right)^2 \chi^2\right). \tag{9}$$

Actually, the procedure yields $-H$, which has a negative spectrum. One way to understand the minus sign is to think about the Wick rotation from Lorentzian signature. Find a complete set of solutions of the equation of motion. For simplicity, work in $d = 3$. Recall that the spherical harmonics $Y_{\ell m}(\theta, \phi)$ form a complete set of solutions of the Laplace equation on a sphere. They have the following properties:

$$\Box_{S^2} Y_{\ell m} = \left(\frac{1}{\sin\theta}\partial_\theta(\sin\theta \, \partial_\theta) + \frac{1}{\sin^2\theta}\partial_\phi^2\right) Y_{\ell m} = -\ell(\ell+1) Y_{\ell m}, \tag{10}$$

$$\sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} Y^*_{\ell m}(\theta, \phi) Y_{\ell m}(\theta', \phi') = \frac{\delta(\theta - \theta')\delta(\phi - \phi')}{\sin\theta}, \tag{11}$$

$$\int d\Omega \, Y_{\ell m}(\theta, \phi) Y^*_{\ell' m'}(\theta, \phi) = \delta_{\ell\ell'}\delta_{mm'}, \tag{12}$$

$$Y^*_{\ell m} = (-1)^m Y_{\ell, -m}. \tag{13}$$

In particular, start with the following ansatz:

$$\chi(\sigma, \theta, \phi) = \sum_{\ell m} b_{\ell m}(\sigma) Y_{\ell m}(\theta, \phi). \tag{14}$$

Show that it is possible to expand the solution in terms of two sets of constant modes, $a^+_{\ell m}$ and $a^-_{\ell m}$, such that

$$H = \sum_{\ell, m} \frac{1}{2}\omega_\ell (a^-_{\ell m} a^+_{\ell m} + a^+_{\ell m} a^-_{\ell m}), \quad \omega_\ell = \ell + \frac{1}{2}. \tag{15}$$

In defining the modes, you should keep in mind that χ is not a real field. Instead, you should impose the following hermiticity property:

$$\chi^\dagger(\tau) = \chi(-\tau). \tag{16}$$

(d) The quantization of the theory follows from imposing the canonical commutation relations:

$$[\chi(\sigma,\theta,\phi), \partial_\sigma \chi(\sigma,\theta',\phi')] = \frac{\delta(\theta-\theta')\delta(\phi-\phi')}{\sin\theta}. \tag{17}$$

The denominator on the right-hand side is present to compensate the factor $\sin\theta$ in the volume element $d\Omega$. The plus sign, as opposed to the usual factor i, is related to the Wick rotation: defining a real time variable $t = -i\sigma$ recovers the missing imaginary unit. Plug the mode expansion into the commutation relations and deduce that the energy spectrum is labeled by n-tuples of positive integers $(\ell_1, \ldots, \ell_n)$, $n = 0, 1, 2, \ldots$:

$$E_{\ell_1,\ldots,\ell_n} = \left(\ell_1 + \frac{1}{2}\right) + \cdots + \left(\ell_n + \frac{1}{2}\right). \tag{18}$$

(e) Compare the spectrum with the scaling dimension of the local operators in the theory. Do they match? The state-operator correspondence is the statement that the answer to this question is affirmative.

(f) (*Optional*) In the usual quantization on constant time slices, the Hamiltonian can be recovered as the integral of T_{00} on a fixed time slice. Consider now the current $j^\mu = -T^{\mu\nu}x_\nu$, where $T^{\mu\nu}$ is the traceless stress-tensor. The expression for the latter in arbitrary dimension is as follows:

$$T_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi - \frac{1}{2}g_{\mu\nu}(\partial\phi)^2 - \xi(\partial_\mu\partial_\nu - g_{\mu\nu}\Box)\phi^2, \quad \xi = \frac{d-2}{4(d-1)}. \tag{19}$$

Compare the Hamiltonian (9) with the flux of the current on a sphere:

$$\int d\Omega\, r^{d-1} x^\mu r j_\mu. \tag{20}$$

If you want, you can keep the parameter ξ generic until the end, and check that this operator and the Hamiltonian have different spectra, unless the stress-tensor is chosen to be traceless.